

The Laplace transform

We have seen that the reason the Fourier transform is so important is that, if we know $\mathcal{F}(f)$, then we know f too (via the Fourier inversion formula).

However, there are functions f , such as the function 1 , or polynomials (such as $f(x) = x^2 \forall x \in \mathbb{R}$)

for which $\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixy} f(y) dy$ is not well-

defined $\left(\begin{array}{l} \text{notice that } \lim_{\substack{a \rightarrow -\infty \\ b \rightarrow +\infty}} \int_a^b e^{-ixy} f(y) dy \\ \text{doesn't exist when } f \text{ is a polynomial.} \end{array} \right)$

It is such situations in which the Laplace transform can be very useful instead!

In particular, for any $f: [0, +\infty) \rightarrow \mathbb{R}$,

we will define the Laplace transform $\mathcal{L}(f)$ of f .

And, once we know $\mathcal{L}(f)$, we will be able to get f . The advantage of $\mathcal{L}(f)$ compared to $\mathcal{F}(f)$

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is that $\mathcal{L}(f)$ makes sense for a much wider class of functions f than $\mathcal{F}(f)$ (and polynomials are such functions, for instance)

The idea is the following:

Let $f: \mathbb{R} \rightarrow \mathbb{C}$. Perhaps f is not in $L^1(\mathbb{R})$... so $\mathcal{F}(f)$ is not well-defined.

However, it is quite likely that for some $x > 0$ (perhaps large), the "perturbation" $e^{-xt} f(t)$, $t \in \mathbb{R}$,

decays so fast as $t \rightarrow +\infty$ that

$e^{-xt} f(t) \in L^1_t(\mathbb{R}, +\infty)$. In that case,

for the function $\phi_x: \mathbb{R} \rightarrow \mathbb{C}$, with

$$\phi_x(t) = \begin{cases} e^{-xt} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases},$$

$\mathcal{F}(\phi_x)$ is actually well-defined. And thus, if we

know $\mathcal{F}(\phi_x)$, we can get ϕ_x by the Fourier inversion formula, and then divide by e^{-xt} to get

$f: [0, +\infty) \rightarrow \mathbb{R}$. So, $\mathcal{F}(\phi_x)$, for this $x > 0$, is a very interesting (useful) function.

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Knowing $\mathcal{F}(\phi_x)(y)$ for all $y \in \mathbb{R}$
(for this fixed $x > 0$)

would give us f . Let's see what $\mathcal{F}(\phi_x)(y)$ is,
for all $y \in \mathbb{R}$:

$$\mathcal{F}(\phi_x)(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iyt} \phi_x(t) dt = \frac{1}{2\pi} \cdot \int_0^{+\infty} e^{-iyt} \phi_x(t) dt =$$

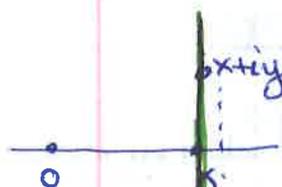
$$\begin{array}{l} \phi_x(t) = 0 \\ \forall t < 0 \end{array}$$

$$= \frac{1}{2\pi} \cdot \int_0^{+\infty} e^{-iyt} \cdot e^{-xt} f(t) dt =$$

$$= \frac{1}{2\pi} \cdot \int_0^{+\infty} e^{-(x+iy)t} f(t) dt$$

We call this $\mathcal{L}(f)(x+iy)$,
the Laplace transform of f
at the complex number $x+iy$.
Notice that, like the Fourier transform,
it is an integral of f against an
exponential. But the exponential now is
not a unit vector: it splits in an extra
 e^{-xt} , which gives f enough decay to
make it integrable, and in the unit
vector e^{-iyt} , which twists the graph
of $e^{-xt} f(t)$, to give its Fourier
transform.

Observe that, by above discussion,



$$\mathcal{L}(f)(x+iy) = \mathcal{L}_x \cdot f(\phi_x)(y), \quad \forall y \in \mathbb{R}.$$

And, knowing $f(\phi_x)$ for this fixed x gives us ϕ_x , and thus f .

So, all we need to get f from $\mathcal{L}(f)$ is to know the values of $\mathcal{L}(f)$ on a vertical line in the complex plane! We just need the values of $\mathcal{L}(f)$ on l_x , rather

(x in the vertical line through x in the complex plane)

than on the whole domain of $\mathcal{L}(f)$.

Let us now properly define the Laplace transform of a function f :

Def: Let $f: [0, +\infty) \rightarrow \mathbb{C}$. Let $x > 0$ be such that the function $\phi_x(t) = e^{-xt} f(t)$, $t \geq 0$, is in $L^1([0, +\infty))$.

Then, for all $y \in \mathbb{R}$, we define

$$\mathcal{L}(f)(x+iy) := \int_0^{+\infty} e^{-(x+iy)t} f(t) dt.$$

In other words, for any $z \in \mathbb{C}$ s.t., for $x = \text{Re}z$, we have $\phi_x \in L^1([0, \infty))$,

we define $\mathcal{L}(f)(z) := \int_0^{\infty} e^{-zt} f(t) dt.$

the union of all vertical lines l_x s.t. $\phi_x \in L^1([0, \infty))$.

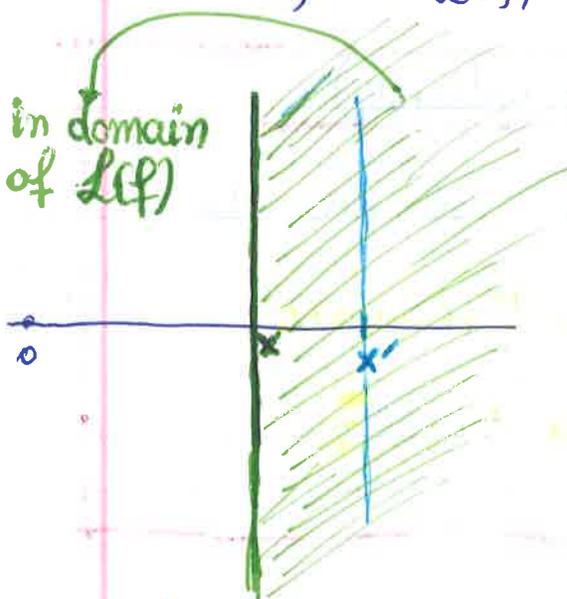
Notice that $\mathcal{L}(f)$ (subset of \mathbb{C}) $\rightarrow \mathbb{C}$.

What does the domain of $\mathcal{L}(f)$ actually look like?

We have explained that, if $\phi_x \in L^1([0, \infty))$, for some fixed x , then $\mathcal{L}(f)(x+iy) (= \mathcal{L}_1 f(\phi_x)(y))$

is well-defined at all $y \in \mathbb{R}$.

Thus, $\mathcal{L}(f)$ is well-defined on the whole vertical line l_x .



Fix such an x . Now, consider $x' > x$. The function $\phi_{x'}(t) = \begin{cases} e^{-x't} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$

decays even faster than ϕ_x , so

$\phi_{x'} \in L^1([0, \infty))$ as well, thus

$\mathcal{L}(f)$ is well-defined on $l_{x'}$ too!

Thus, $\mathcal{L}(f)$ is well-defined on any vertical line on the right of l_x , so on the whole green shadowed area above.

Note that, potentially, there may exist no $x > 0$ s.t. $\phi_x \in L^1([0, \infty))$ (such as when $f(t) = e^{t^2}$; no exponential e^{-xt} can cancel its fast growth).

But it is anyway quite likely that some such $x > 0$ exists. There exists such x for polynomials f , for instance. So, the Laplace transform of any polynomial is well-defined (actually, on the whole complex half-plane , without the boundary).

To find f from $\mathcal{L}(f)$, all we need to know is $\mathcal{L}(f)$ on some vertical line l_x (rather than on its whole domain). Let's see this:

Laplace inversion formula (Bromwich integral):

Let $f: [0, \infty) \rightarrow \mathbb{C}$, and let $x > 0$ be a fixed real number, s.t. $\phi_x \in L^1([0, \infty))$.

Then, $f(t) = \frac{1}{2\pi i} \int_{l_x} e^{tz} \mathcal{L}(f)(z) dz$, $\forall t \in [0, \infty)$

Proof: As we have already seen,

$$\mathcal{L}(f) \left(\underbrace{x+iy}_{\in l_x} \right) = 2\pi \cdot f(\phi_x)(y), \quad \forall y \in \mathbb{R}.$$

And : $\forall t \in [0, +\infty)$, by Fourier inversion for ϕ_x (remember, $\phi_x(t) = e^{-xt} f(t)$, $\forall t \in [0, +\infty)$),

we obtain:

$$\begin{aligned} \phi_x(t) &= \int_{-\infty}^{+\infty} f(\phi_x)(y) e^{ity} dy = \\ &= \int_{-\infty}^{+\infty} \frac{1}{2\pi} \cdot \mathcal{L}(f)(x+iy) e^{ity} dy \quad \frac{\text{multiply and divide}}{\text{by } e^{-xt}} \\ &= \frac{e^{-xt}}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{ity} e^{xt} \mathcal{L}(f)(x+iy) dy = \\ &= \frac{e^{-xt}}{2\pi} \int_{-\infty}^{+\infty} e^{(x+iy)t} \mathcal{L}(f)(x+iy) dy \quad \frac{\begin{matrix} e^{(-\infty, +\infty)} \\ x+iy \text{ is the} \\ \text{parametrisation} \\ \text{of the curve} \\ \Gamma_x \end{matrix}} \end{aligned}$$

$$\begin{aligned} \underline{\underline{x+iy=z}} \\ \underline{\underline{dz = d(x+iy) = i dy}} \quad \frac{e^{-xt}}{2\pi} \cdot \int_{\Gamma_x} e^{zt} \mathcal{L}(f)(z) \frac{dz}{i} = \\ = \frac{e^{-xt}}{2\pi i} \cdot \int_{\Gamma_x} e^{zt} \mathcal{L}(f)(z) dz. \end{aligned}$$

Since $\phi_x(t) = e^{-xt} f(t) \forall t \geq 0$, we have :

$$f(t) = \frac{\phi_x(t)}{e^{-xt}} = \frac{1}{2\pi i} \int_{\Gamma_x} e^{tz} \mathcal{L}(f)(z) dz, \quad \forall t \geq 0.$$





The Laplace transform is very useful when solving differential equations, as it makes derivatives disappear (like the Fourier transform):

→ Let $f: [0, \infty) \rightarrow \mathbb{C}$ be continuously differentiable
↓
so that we can do integration by parts for f' .

Let $z \in \mathbb{C}$, with $z = x + iy \in \mathbb{R}$, s.t. $\phi_x \in L^1([0, \infty))$
↳ $\phi_x(t) = e^{-xt} f(t), t \geq 0$.

Then, $\mathcal{L}(f')(z) = z \cdot \mathcal{L}(f)(z) - f(0)$

↓
the initial condition for $t=0$.

(So, the Laplace transform turns one derivative into multiplication, translated by a constant $f(0)$).

Proof: $\mathcal{L}(f')(z) = \int_0^{\infty} e^{-zt} f'(t) dt$ ↳ integration by parts

$= [e^{-zt} f(t)]_{t=0}^{t=\infty} - \int_0^{\infty} (e^{-zt})' f(t) dt.$

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$$\bullet \int_0^{+\infty} (e^{-zt})' f(t) dt = (-z) \cdot \underbrace{\int_0^{+\infty} e^{-zt} f(t) dt}_{\text{well-defined at this } z} = (-z) \cdot \mathcal{L}(f)(z).$$

$$\bullet [e^{-zt} f(t)]_{t=0}^{t=+\infty} = \lim_{t \rightarrow +\infty} \underbrace{(e^{-zt} f(t))}_{\substack{\text{unit} \\ \text{vector}}} - \underbrace{e^{-z \cdot 0} f(0)}_{f(0)} =$$

$$= e^{-(x+iy)t} f(t) = \underbrace{e^{-iyt}}_{\text{unit vector}} \cdot \underbrace{e^{-xt} f(t)}_{\substack{\phi_x \in L^1([0, +\infty)), \\ \text{so goes to } 0 \\ \text{as } t \rightarrow +\infty}}$$

↓
0

$$= 0 - f(0)$$

$$\text{So, } \mathcal{L}(f')(z) = z \cdot \mathcal{L}(f)(z) - f(0).$$



Careful when calculating the Laplace transform of $f^{(k)}$, for $k > 1$: we get $z^k \mathcal{L}(f)(z)$, translated by a polynomial. For instance:

$$\begin{aligned} \mathcal{L}(f'') &= z \cdot \mathcal{L}(f') - f'(0) = z \cdot (z \cdot \mathcal{L}(f)(z) - f(0)) - f'(0) = \\ &= z^2 \mathcal{L}(f)(z) - \underbrace{z \cdot f(0) - f'(0)}_{\text{polynomial depending on initial conditions for all lower derivatives of } f}. \end{aligned}$$

polynomial depending on initial conditions for all lower derivatives of f .

Lecture 32:

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Usefulness of the Laplace transform:

The Laplace transform is very useful when solving differential equations (partial or ordinary) as, like the Fourier transform, it makes derivatives disappear.

The Laplace transform is sometimes better to use than the Fourier transform actually, as there are more functions f with well-defined Laplace transform, than with well-defined Fourier transform. This means that:

1) The Laplace transform can reveal more solutions than the Fourier transform.

ex: $f'(t) = 0, \forall t \in \mathbb{R}$ or ≥ 0 .

We know that all constant functions are solutions to this equation.

But the Fourier transform will only give the solution 0, as this is the only constant whose Fourier transform exists. Indeed:

the Fourier transform will give all f solutions, s.t. f exists.

$$f'(t) = 0 \quad \forall t \quad \xrightarrow{\text{assuming that } f \text{ exists}} \quad \hat{f}(t) = \hat{0} = 0 \quad \forall t$$

$$\Rightarrow \hat{f}(t) = 0 \quad \forall t$$

$$\Rightarrow f(t) = 0 \quad \forall t.$$

the Laplace transform will give all sols f for which $L(f)$ exists. And that is much more likely than f existing.

But: $f'(t) = 0 \quad \forall t \quad \xrightarrow{\text{assuming that } L(f) \text{ exists}}$

$$\Rightarrow z L(f)(z) - f(0) = L(0) = 0, \quad \forall z \text{ appropriate}$$

$$\Rightarrow z L(f)(z) = f(0), \quad \forall z \text{ appropriate}$$

$$\Rightarrow L(f)(z) = f(0) \cdot \frac{1}{z}, \quad \forall z \text{ appropriate}$$

$$\Rightarrow \dots \Rightarrow f(t) = f(0), \quad \forall t \geq 0.$$

This way we have gotten all solutions to $f'(t) = 0, \forall t \geq 0$.

② The Laplace transform can give easily solutions to differential equations with constant terms, as the Laplace transform of constants exists. or polynomial! ②

ex: $f''(t) = c \quad \forall t (\in \mathbb{R} \text{ or } \geq 0)$.

\hat{c} is not a function, so taking Fourier transforms on both sides isn't pretty.

But $\mathcal{L}(c)$ exists. So, taking Laplace transforms on both sides is meaningful. It will give $\mathcal{L}(f)$, and therefore f .

→ * Notice that in a PDE, where we are looking for a solution $u(x,t)$ w.r.t. a spatial variable x and a time variable t , it makes sense:

- to apply the Fourier transform w.r.t. x
- or - to apply the Laplace transform w.r.t. t .

The reason is that the Laplace transform

acts on functions $f: [0, \infty) \rightarrow \mathbb{R}$, i.e. with a non-negative variable; and the non-negative variable in a PDE is usually time t , not

the spatial variable x .

→ in more detail than in textbook.

→ Example 1, p. 440 of the textbook:

Solve $f''(t) + 4f'(t) + 4f(t) = t^2 e^{-2t}$, $\forall t \geq 0$,

with initial conditions $f(0) = f'(0) = 0$.

Solution: Let $z \in \mathbb{C}$, st. $\text{Re}z > x_0$, where x_0 is st.

$e^{-x_0 t} \cdot (t^2 e^{-2t}) \in L^1_t([0, \infty))$

($x_0 = 0$ will do; even $x_0 > -2$ will do, but it doesn't matter; all we need is $L(f)$ on a vertical line alone).

Then, $\mathcal{L}(\text{RHS})$ exists, so $\mathcal{L}(\text{LHS})$ exists. And:

• $\mathcal{L}(\text{LHS})(z) = \mathcal{L}(f'' + 4f' + 4f)(z)$ \mathcal{L} linear
+ assuming that $\mathcal{L}(f'')$, $\mathcal{L}(f')$, $\mathcal{L}(f)$ all exist (so we will only get solutions f with this property)

$= \mathcal{L}(f'')(z) + 4\mathcal{L}(f')(z) + 4\mathcal{L}(f)(z) =$

$= \underbrace{z^2 \cdot \mathcal{L}(f)(z) - z \cancel{f'(0)} - \cancel{f(0)}}_0 + 4 \cdot \underbrace{(z \cdot \mathcal{L}(f)(z) - \cancel{f(0)})}_0$

$+ 4 \cdot \mathcal{L}(f)(z) =$

$= \mathcal{L}(f)(z) \cdot (z^2 + 4z + 4) = \mathcal{L}(f)(z) \cdot (z+2)^2$

• $\mathcal{L}(\text{RHS})(z) = \mathcal{L}(t^2 e^{-2t})(z) = \int_0^{\infty} e^{-zt} t^2 e^{-2t} dt = \int_0^{\infty} e^{-(z+2)t} t^2 dt =$

$= \int_0^{\infty} \frac{(e^{-(z+2)t})'}{-(z+2)} t^2 dt = -\frac{1}{z+2} \cdot [e^{-(z+2)t} t^2]_0^{\infty} + \frac{1}{z+2} \cdot \int_0^{\infty} e^{-(z+2)t} \cdot 2t dt =$

$$= -\frac{1}{z+2} \cdot \left(\lim_{t \rightarrow +\infty} e^{-(z+2)t} t^2 - \cancel{e^{-(z+2) \cdot 0} \cdot 0^2} \right) + \frac{2}{z+2} \cdot \int_0^{+\infty} e^{-(z+2)t} t dt.$$

We will now show that $\lim_{t \rightarrow +\infty} e^{-(z+2)t} t^2$,

thanks to our choice of z . Indeed, $z = x + iy \in \mathbb{C}$,

so $e^{-(z+2)t} t^2 = e^{-(x+2+iy)t} t^2 =$

$$= \underbrace{e^{-(x+2)t} t^2}_{\text{decays faster than } e^{-(x_0+2)t} t^2} \cdot \underbrace{e^{-iyt}}_{\text{unit vector}} \xrightarrow{t \rightarrow +\infty} 0$$

as $\text{Re}z (= x) > x_0$;
and $e^{-(x_0+2)t} t^2 =$

$$= e^{-x_0 t} \cdot (e^{-2t} t^2) \xrightarrow{t \rightarrow +\infty} 0,$$

as it is in $L^1([0, +\infty))$
(that's how we picked x_0).

So, $e^{-(x+2)t} t^2 \xrightarrow{t \rightarrow +\infty} 0$.

So, $\mathcal{L}(\text{RHS}) = \frac{2}{z+2} \cdot \int_0^{+\infty} e^{-(z+2)t} t dt \stackrel{\text{similarly}}{=} \frac{2}{(z+2)^3}$.

So, $\mathcal{L}(f)(z) \cdot (z+2)^2 = \frac{2}{(z+2)^3}, \forall z \in \mathbb{C} \text{ with } \text{Re}z > x_0$

$\Rightarrow \mathcal{L}(f)(z) = \frac{2}{(z+2)^5}, \forall z \in \mathbb{C} \text{ with } \text{Re}z > x_0.$

There are tables in p. 469-471 of the textbook, that give

us f if we know $\mathcal{L}(f)$, in some standard cases

Our case is such a case: $\mathcal{L}(f)(z) = \frac{2}{(z+2)^5}$

implies that $f(t) = \frac{t^4 \cdot e^{-2t}}{12}$, $\forall t \geq 0$,

according to the tables.

(Then, technically, we have to make sure that this f is the unique solution; or, else, our assumption that $\mathcal{L}(f), \mathcal{L}(f'), \mathcal{L}(f'')$ exist cost us solutions...

→ How to calculate f from $\mathcal{L}(f)$ via residue theory:

p. 697 of text book

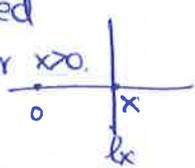
When we are not lucky enough to find our $\mathcal{L}(f)$ in a table, then we have to calculate $f(t), \forall t \geq 0$, via the Laplace inversion formula.

One thing that may help us here is residue theory (and the example above falls in this category of $\mathcal{L}(f)$'s, whose inversion formula can be understood via residue theory).

Let us start with the special case

$\mathcal{L}(f)(z) = \frac{1}{z^2}$, say $\forall z \in \Gamma_x$, for a fixed vertical line Γ_x in \mathbb{C} , for $x > 0$.

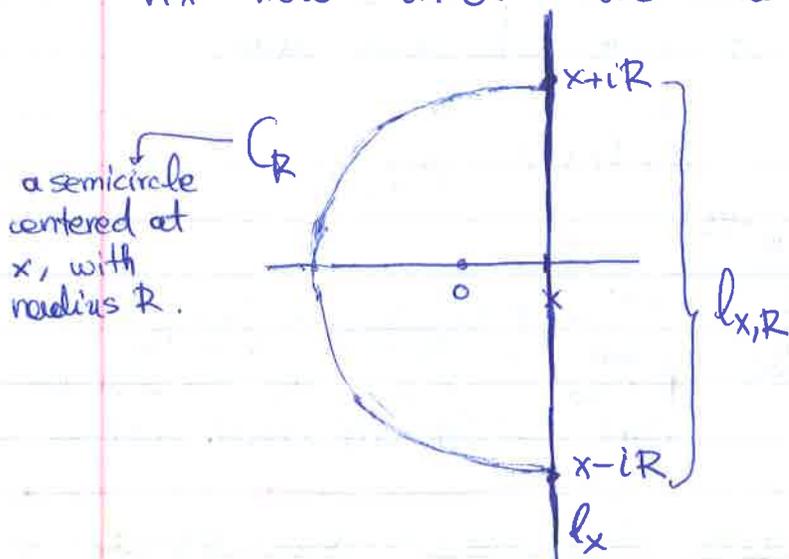
(we don't care about the formula of $\mathcal{L}(f)$ anywhere else)



We know that $f(t) = \frac{1}{2\pi i} \cdot \int_{\Gamma_x} \left(\frac{1}{z^2} \right)^{\mathcal{L}(f)(z)} \cdot e^{zt} dz$, $\forall t \geq 0$.

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Fix now $t \geq 0$. We will evaluate $f(t)$.



We know that

$$f(t) = \frac{1}{2\pi i} \cdot \int_{l_x} \frac{1}{z^2} e^{zt} dz,$$
 for this fixed t .

And:

$$\int_{l_x} \frac{1}{z^2} e^{zt} dz = \lim_{R \rightarrow \infty} \int_{l_{x,R}} \frac{1}{z^2} e^{zt} dz.$$

So, we need to understand $\int_{l_{x,R}} \frac{1}{z^2} e^{zt} dz$ for large R .

We make $l_{x,R}$ part of a closed loop, as above. Then,

$$\int_{l_{x,R}} \frac{1}{z^2} e^{zt} dz + \int_{C_R} \frac{1}{z^2} e^{zt} dz = 2\pi i \cdot \sum_{\substack{a \text{ the} \\ \text{singularities} \\ \text{of } \frac{e^{zt}}{z^2} \\ \text{surrounded} \\ \text{by } C_R \cup l_{x,R}}} \text{Res} \left(\frac{e^{zt}}{z^2}; a \right).$$

We were able to apply the residue theorem as there is only one (finite number) singularity of $\frac{e^{zt}}{z^2} : 0$

And it is surrounded by our loop, (when R is large) it is not on the loop.

So:

$$\int_{l_{x,R}} \frac{1}{z^2} e^{zt} dz + \int_{C_R} \frac{1}{z^2} e^{zt} dz = 2\pi i \cdot \text{Res} \left(\frac{e^{zt}}{z^2}; 0 \right),$$

for all R large.

0 is a pole of order 2 for $\frac{e^{zt}}{z^2}$. In fact:

$$\frac{e^{zt}}{z^2} = \frac{1 + (zt) + \frac{(zt)^2}{2!} + \frac{(zt)^3}{3!} + \dots}{z^2} = \frac{1}{z^2} + \left(\frac{t}{z}\right) + \frac{t^2}{2!} + \frac{t^3}{3!}z + \dots,$$

so $\text{Res}\left(\frac{e^{zt}}{z^2}; 0\right) = t$. Thus, for all large R :

$$\int_{\Gamma_{x,R}} \frac{1}{z^2} e^{zt} dz + \int_{\mathbb{C}_R} \frac{1}{z^2} e^{zt} dz = 2\pi i \cdot t.$$

As $R \rightarrow +\infty$:

- $\int_{\Gamma_{x,R}} \frac{1}{z^2} e^{zt} dz \xrightarrow{R \rightarrow +\infty} \int_{\Gamma_x} \frac{1}{z^2} e^{zt} dz$, our desired integral.
- $\int_{\mathbb{C}_R} \frac{1}{z^2} e^{zt} dz \xrightarrow{R \rightarrow +\infty} 0$ (and this wouldn't be necessarily

true if we were looking at the semicircle $\Gamma_{x,R}$ on the right of x , rather than the one on the left!!

Indeed: $C_R = \{x + Re^{i\theta} : \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$. So:

$$\int_{C_R} \frac{1}{z^2} e^{zt} dz = \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} \frac{1}{(x+Re^{i\theta})^2} e^{(x+Re^{i\theta})t} Rie^{i\theta} d\theta$$

$= e^{xt} \cdot e^{R(\cos\theta + i\sin\theta)t}$
 $= e^{xt} \cdot e^{R(\cos\theta)t} \cdot \underbrace{e^{iR(\sin\theta)t}}_{\text{unit vector}}$

$$\text{So, } \left| \int_{C_R} \frac{1}{z^2} e^{zt} dz \right| \leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{R}{|x+Re^{i\theta}|^2} \cdot e^{xt} \cdot e^{R(\cos\theta)t} d\theta =$$

$$= e^{xt} \cdot \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{R}{|x+Re^{i\theta}|^2} e^{R(\cos\theta)t} d\theta.$$

Note that x, t are fixed. So, as R varies:

- e^{xt} is a constant.
- $|x + Re^{i\theta}|^2 \sim R^2$ (our R is large)
- $\cos\theta \leq 0$ for $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

$$\Rightarrow \underbrace{R(\cos\theta) \cdot t}_{\substack{t > 0 \\ R > 0}} \leq 0 \Rightarrow \underline{e^{R(\cos\theta)t} \leq 1}$$

↳ if we had picked as C_R the semicircle on the right of x , then θ would be in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so $R(\cos\theta)t$ would be taking huge values! That's why we get the left C_R .

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Thus: for large R ,

$$\left| \int_{C_R} \frac{1}{z^2} e^{zt} dz \right| \leq (\text{constant}) \cdot \int_{\pi/2}^{3\pi/2} \frac{1}{R^2} dt \leq$$

$$\leq (\text{constant}) \cdot \frac{1}{R^2} \xrightarrow{R \rightarrow +\infty} 0$$

$$\Rightarrow \int_{C_R} \frac{1}{z^2} e^{zt} dz \xrightarrow{R \rightarrow +\infty} 0$$

Eventually:

$$\lim_{R \rightarrow +\infty} \left(\int_{L_{x,R}} \frac{1}{z^2} e^{zt} dz + \int_{C_R} \frac{1}{z^2} e^{zt} dz \right) = 2\pi i t$$

$$\lim_{R \rightarrow +\infty} \int_{L_{x,R}} \frac{1}{z^2} e^{zt} dt + \lim_{R \rightarrow +\infty} \int_{C_R} \frac{1}{z^2} e^{zt} dz$$

$$= \int_{L_x} \frac{1}{z^2} e^{zt} dt + 0$$

$$\text{So, } \int_{L_x} \frac{1}{z^2} e^{zt} dt = 2\pi i t \Rightarrow \underbrace{\frac{1}{2\pi i} \int_{L_x} \frac{1}{z^2} e^{zt} dt}_{= f(t)} = t$$

$$\text{So, } \boxed{f(t) = t, \forall t \geq 0.} \text{ (verify from a table)}$$

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In fact, notice that we could have worked the same way if, instead of $\mathcal{L}(f)(z) = \frac{1}{z^2}$, we had $\mathcal{L}(f)(z) = \frac{P(z)}{Q(z)}$, where P, Q polynomials, with $\deg Q \geq \deg P + 2$.

In addition to this, notice that we didn't exploit at all the extra decay we get from

$e^{R(\cos \theta)t}$, due to Jordan's lemma! The existence of $e^{R(\cos \theta)t}$ in the integral gives an extra

decay of order $\frac{1}{R}$, because

$$\int_{\theta=0}^{3\pi/2} e^{R(\cos \theta)t} d\theta = \int_{\theta=\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{R(\sin(\frac{\pi}{2}-\theta))t} d\theta = \int_{\tilde{\theta}=\frac{\pi}{2}}^{\pi} e^{-R(\sin \tilde{\theta})t} d\tilde{\theta} = O\left(\frac{1}{R}\right).$$

So, we would have been OK even if we had started with $\mathcal{L}(f)(z) = \frac{P(z)}{Q(z)}$, where P, Q polynomials, with

$$\deg Q \geq \deg P + 1.$$

(for instance, $\mathcal{L}(f)(z) = \frac{1}{z}$.)

So, if $\mathcal{L}(f)(z) = \frac{P(z)}{Q(z)}$, where P, Q polynomials with $\deg Q \geq \deg P + 1$

on some line l_x in the complex plane
(vertical line through x)

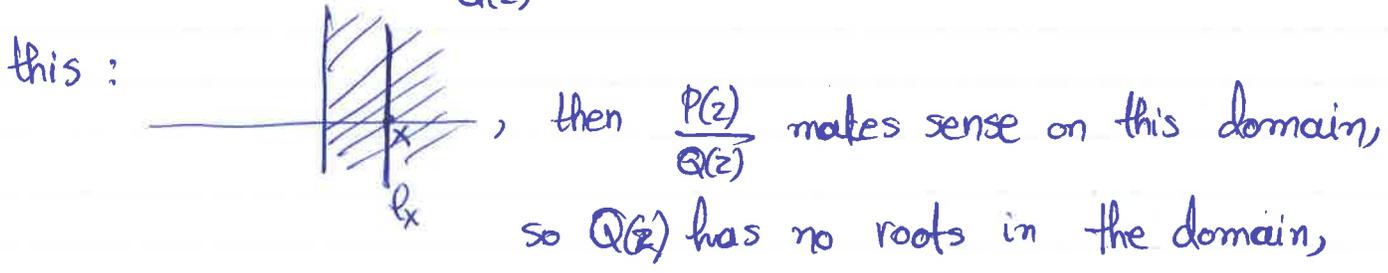
(and even if we don't know $\mathcal{L}(f)(z)$ for any z outside the line l_x),

then $f(t) = \sum \text{Res} \left(\frac{P(z)}{Q(z)} \cdot e^{zt} ; a \right)$
a the singularities of on the left of l_x

(as long as of course there are no singularities on l_x itself; otherwise we wouldn't have been able to apply the residue theorem).

(Note that, if l_x is in the domain of $\mathcal{L}(f)$, then $l_{x'}$ is also in the domain of $\mathcal{L}(f)$, for all $x' > x$.)

So, if $\mathcal{L}(f)(z) = \frac{P(z)}{Q(z)}$ for all z in a domain like



(12)

thus all the singularities of $\frac{P(z)}{Q(z)} e^{zt}$ are on the left of the domain. Thus, using a different γ'

won't change anything; we are still looking at the same singularities. I.e., the singularities on the left of

γ' are the same as the singularities on the left of γ . They are, in fact, all the singularities of $\frac{P(z)}{Q(z)} e^{zt}$.)